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THE OPTIMUM RECTILINEAR MOTION OF A TWO-MASS SYSTEM†

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The forward rectilinear motion of a system of two rigid bodies along a horizontal plane is considered. Forces of dry friction act between the bodies and the plane, and the motion is controlled by internal forces of interaction between the bodies. A periodic motion in which the system moves along a straight line is constructed. The optimum parameters of the system and a control law are found corresponding to the maximum mean velocity of motion of the system as a whole.

1. INTRODUCTION

Before giving a formal statement of the problem, we will discuss the possibilities of a system moving under the action of internal forces in the presence of dry friction.

Consider a system of N identical rigid bodies of mass m on a horizontal plane (Fig. 1). Forces of dry friction, obeying Coulomb's law, act between the bodies and the plane; the coefficient of friction of the bodies against the plane is denoted by k. Adjacent bodies may interact with one another via internal forces F. Under certain conditions the system may move along a straight line x.

We will describe one simple mode of motion. At the initial instant all the masses are at rest. First, mass 1 moves along the x axis for a certain distance Δx, less than the distance between masses 1 and 2, and then stops. During this time, all the other masses are stationary. The motion of mass 1 consists of an accelerating phase, in which the interaction force F between masses 1 and 2 exceeds the friction force ($F > mgk$), and a decelerating phase, in which $F < mgk$, where g is the acceleration due to gravity. In the meantime the other masses must interact with one another and remain in a state of rest, with the condition that the inequality $|F| \leqslant (N-1)mgk$ holds throughout the motion of mass 1. Next, mass 2 moves through a distance Δx in exactly the same way as mass 1 did before, all the other masses remaining stationary. In the process, the sum of the forces that the adjacent stationary masses 1 and 3 exert on mass 2 must be greater than mgk in the accelerating phase and less than mgk in the decelerating phase. Continuing the process of the masses moving in turn, as described, we end it with mass N moving through a distance Δx. As a result, the whole system is displaced a distance Δx, after which the process may be repeated any number of times. The periodic motion just described is possible if the force with which two adjacent masses interact varies within the limits

$$mgk < |F| \leqslant (N-1)mgk \tag{1.1}$$

It should be mentioned that wave motions of the bodies in the presence of dry friction forces were considered in [1, 2]. It follows from inequalities (1.1) that the method of movement just described is applicable only when $N \geqslant 3$. It is therefore of interest to investigate the possibilities of motion in a simple two-mass system.

2. STATEMENT OF THE PROBLEM

Consider a system of two interacting rigid bodies of masses m_1 and m_2, capable of moving along the x axis on a horizontal plane (Fig. 2). Dry friction forces obeying Coulomb's law act between the bodies and the plane; the coefficients of friction for masses, m_1 and m_2 are k_1 and k_2, respectively. The force that mass m_2 exerts on mass m_1 is denoted by F; then mass m_1 exerts a force $-F$ on mass m_2. We shall assume that the distance L between the masses may vary in the interval $L_0 - \eta \leqslant L \leqslant L_0$, where L_0 is the initial distance and $\eta \leqslant L_0$ is the admissible distance to within which the masses may approach one another compared with the initial state, that is, the path or range of admissible relative motion.

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Fig. 1

Let x_i denote the displacement of mass m_i from its initial position, and v_i its velocity, $i = 1, 2$. The distance between the masses if $L_0 + x_2 - x_1$. We have

$$\dot{x}_i = v_i, \quad i = 1, 2 \tag{2.1}$$

The equations of motion of the masses m_1 and m_2, taking Coulomb's law into account, may be written in the form

$$\begin{aligned} m_1\dot{v}_1 &= F - m_1 g k_1 \operatorname{sign} v_1 \quad \text{for} \quad v_1 \neq 0, \quad |F| \leq m_1 g k_1 \quad \text{for} \quad v_1 = 0 \\ m_2\dot{v}_2 &= -F - m_2 g k_2 \operatorname{sign} v_2 \quad \text{for} \quad v_2 \neq 0, \quad |F| \leq m_2 g k_2 \quad \text{for} \quad v_2 = 0 \end{aligned} \tag{2.2}$$

Suppose at the initial instant $t = 0$ both masses are at rest and at the maximum admissible distance L_0 from one another.

It is required to construct a piecewise-constant law for the variations of the control force $F(t)$, under which both masses will move the same distance ξ in a time T and be again at rest at the end of the motion. Throughout, the distance between the masses must remain within the given limits, as can be expressed by the inequality

$$0 \leq x_1(t) - x_2(t) \leq \eta, \quad t \in [0, T] \tag{2.3}$$

where $\eta > 0$ is the given constant. The required motion will consist of a sequence of four steps, as follows:

1. Forward accelerated motion of mass m_1, mass m_2 remains stationary.
2. Forward decelerated motion of mass m_1, forward accelerated motion of mass m_2.
3. Reverse accelerated motion of mass m_1, forward accelerated motion of mass m_2.
4. Reverse decelerated motion of mass m_1, forward decelerated motion of mass m_2.

The meaning of these steps needs some explanation. Let us consider the mass m_2 as the primary mass and the mass m_2 as the smaller, secondary one ($m_1 < m_2$). Since the masses are initially at their greatest admissible distance from one another, they will approach one another in the first step, but in such a way as not to produce reverse motion of mass m_2. The main forward advance of mass m_2 will occur in the second and third steps, because of the strong mutual repulsion of the masses. In the fourth step, the masses slow down in such a way that they halt simultaneously, at the same distance from one another as initially. Note that the primary mass m_2, unlike the secondary one, does not move backwards. This scheme of motion therefore seems quite rational. In what follows we will assume that $m_1 < m_2$.

Let τ_i denote the durations of the steps and let F_i be the constant values of the control force at these steps, $i = 1, 2, 3, 4$. The only difference between steps 2 and 3 is the change in the sign of the velocity (but not the acceleration) of one mass, m_1. To simplify matters, therefore, we shall assume that the control force remains unchanged during these steps: $F_2 = F_3$. For the motion to possess these properties, the constants F_i must satisfy the inequalities

$$m_1 g k_1 < F_1 < m_2 g k_2, \quad F_2 < -m_2 g k_2, \quad F_4 > -m_1 g k_1 \tag{2.4}$$

For the required motion to satisfy all the stipulated conditions, the parameters must obey several relations. To determine these relations, we shall compute all the steps of the motion.

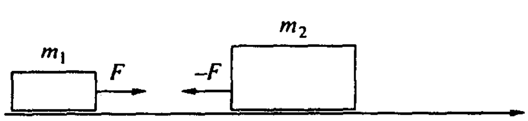

Fig. 2

3. COMPUTATION OF THE MOTION

Let us integrate Eqs (2.1) and (2.2) successively over all steps of the motion, taking inequalities (2.4) into account and assuming that the coordinates x_1, x_2 and the velocities υ_1, υ_2 satisfy continuity conditions at the boundaries of the steps. At the instant $t = 0$ we have the initial conditions

$$x_1(0) = x_2(0) = 0, \quad \upsilon_1(0) = \upsilon_2(0) = 0 \tag{3.1}$$

For the first step we obtain, noting conditions (3.1)

$$\begin{aligned} &\upsilon_1(t) = (F_1 m_1^{-1} - gk_1)t, \quad x_1(t) = (F_1 m_1^{-1} - gk_1)t^2/2 \\ &\upsilon_2(t) = 0, \quad x_2(t) = 0, \quad t \in [0, \ \tau_1] \end{aligned} \tag{3.2}$$

For the second step we have

$$\begin{aligned} &\upsilon_1(t) = (F_2 m_1^{-1} - gk_1)(t - \tau_1) + (F_1 m_1^{-1} - gk_1)\tau_1 \\ &x_1(t) = (F_2 m_1^{-1} - gk_1)(t - \tau_1)^2/2 + (F_1 m_1^{-1} - gk_1)\tau_1(t - \tau_1) + (F_1 m_1^{-1} - gk_1)\tau_1^2/2 \\ &\upsilon_2(t) = -(F_2 m_2^{-1} + gk_2)(t - \tau_1) \\ &x_2(t) = -(F_2 m_2^{-1} + gk_2)(t - \tau_1)^2/2, \quad t \in [\tau_1, \ \tau_1 + \tau_2] \end{aligned} \tag{3.3}$$

Since the velocity υ_1 must vanish at the end of the second step, we have the condition

$$\upsilon_1(\tau_1 + \tau_2) = (F_2 m_1^{-1} - gk_1)\tau_2 + (F_1 m_1^{-1} - gk_1)\tau_1 = 0 \tag{3.4}$$

Integrating Eqs (2.1) and (2.2) over the third step and satisfying condition (3.4), in addition to the continuity of the coordinates and velocities, we obtain

$$\begin{aligned} &\upsilon_1(t) = (F_2 m_1^{-1} + gk_1)(t - \tau_1 - \tau_2) \\ &x_1(t) = (F_2 m_1^{-1} + gk_1)(t - \tau_1 - \tau_2)^2 + (F_2 m_1^{-1} - gk_1)\tau_2^2/2 + (F_1 m_1^{-1} - gk_1)\tau_1(\tau_2 + \tau_1/2) \\ &\upsilon_2(t) = -(F_2 m_2^{-1} + gk_2)(t - \tau_1) \\ &x_2(t) = -(F_2 m_2^{-1} + gk_2)(t - \tau_1)^2/2, \quad t \in [\tau_1 + \tau_2, \ \tau_1 + \tau_2 + \tau_3] \end{aligned} \tag{3.5}$$

Finally, for the fourth step we have

$$\begin{aligned} &\upsilon_1(t) = (F_4 m_1^{-1} + gk_1)(t - \tau_1 - \tau_2 - \tau_3) + (F_2 m_1^{-1} + gk_1)\tau_3 \\ &x_1(t) = (F_4 m_1^{-1} + gk_1)(t - \tau_1 - \tau_2 - \tau_3)^2/2 + (F_2 m_1^{-1} + gk_1)\tau_3(t - \tau_1 - \tau_2 - \tau_3) + \\ &+(F_2 m_1^{-1} + gk_1)\tau_3^2/2 + (F_2 m_1^{-1} - gk_1)\tau_2^2/2 + (F_1 m_1^{-1} - gk_1)\tau_1(\tau_2 + \tau_1/2) \\ &\upsilon_2(t) = -(F_4 m_2^{-1} + gk_2)(t - \tau_1 - \tau_2 - \tau_3) - (F_2 m_2^{-1} + gk_2)(\tau_2 + \tau_3) \\ &x_2(t) = -(F_4 m_2^{-1} + gk_2)(t - \tau_1 - \tau_2 - \tau_3)^2/2 - (F_2 m_2^{-1} + gk_2)(\tau_2 + \tau_3)^2/2 - \\ &-(F_2 m_2^{-1} + gk_2)(\tau_2 + \tau_3)(t - \tau_1 - \tau_2 - \tau_3), \quad t \in [\tau_1 + \tau_2 + \tau_3, \ T] \end{aligned} \tag{3.6}$$

The duration T of the entire motion is expressed as

$$T = \tau_1 + \tau_2 + \tau_3 + \tau_4 \tag{3.7}$$

At the end of the motion, the velocities of both bodies must equal zero, and their displacements should equal ξ

$$\upsilon_1(T) = \upsilon_2(T) = 0, \quad x_1(T) = x_2(T) = \xi \tag{3.8}$$

Formulae (3.2), (3.3), (3.5) and (3.6) completely describe the entire motion. The parameters F_1, F_2, F_4 and τ_i (i = 1, 2, 3, 4) introduced above must satisfy conditions (3.4), (3.7) and (3.8).

To simplify the subsequent arguments, we will introduce the following dimensionless parameters

$$F_1 = m_1 g k_1 f_1, \quad F_2 = -m_1 g k_1 f, \quad F_4 = m_1 g k_1 f_4$$
$$\nu = m_2 k_2 (m_1 k_1)^{-1}, \quad \mu = m_1 / m_2 < 1 \tag{3.9}$$

In terms of the parameters f_1, f_2, f_4 and ν, inequalities (2.4) become

$$1 < f_1 < \nu, \quad f > \nu, \quad f_4 > -1 \tag{3.10}$$

After substituting (3.9), condition (3.4) becomes

$$(f_1 - 1)\tau_1 - (f+1)\tau_2 = 0 \tag{3.11}$$

Substituting solutions (3.6) for v_1 and v_2 into the first two conditions of (3.8) and simplifying, using Eqs (3.7) and (3.9), we obtain

$$(f_4 + 1)\tau_4 - (f-1)\tau_3 = 0$$
$$-(f_4 + \nu)\tau_4 + (f - \nu)(\tau_2 + \tau_3) = 0 \tag{3.12}$$

The three linear equations (3.11) and (3.12) may be used to express the durations of all the steps in terms of τ_1:

$$\tau_2 = \frac{f_1 - 1}{f+1}\tau_1, \quad \tau_3 = \frac{(f-\nu)(f_1 - 1)(f_4 + 1)}{(f+1)(f+f_4)(\nu - 1)}\tau_1$$
$$\tau_4 = \frac{(f-1)(f-\nu)(f_1 - 1)}{(f+1)(f+f_4)(\nu-1)}\tau_1 \tag{3.13}$$

Substituting (3.13) into (3.7), we express τ_1 in terms of T:

$$\tau_1 = \frac{(f+1)(\nu-1)T}{f(f_1 - 1) + f(\nu - 1) + \nu - f_1} \tag{3.14}$$

Note that the numerators and denominators of all formulae (3.13) and (3.14) are positive, by virtue of inequalities (3.10).

We now substitute solutions (3.6) for x_1 and x_2 into the last two conditions of (3.8). After simplifying, using relations (3.7), (3.9) and (3.13), we obtain

$$\xi = A(f_1 - 1)[(f+f_4)(f+f_1)(f+1)(\nu-1)^2 - (f_4+1)(f_1-1)(f-1)(f-\nu)^2]$$
$$\xi = A\mu(f_4 + \nu)(f_1 - 1)^2(f-1)^2(f-\nu) \tag{3.15}$$
$$A = (gk_1\tau_1^2/2)(f+f_4)^{-1}(f+1)^{-2}(\nu-1)^{-2}$$

The motion must also satisfy condition (2.3). It follows from the description of steps 1–4 in Section 2 that the masses m_1 and m_2 approach one another in step 1 but draw apart in steps 3 and 4. Consequently, they are at their least distance apart in step 2. Let us find that least distance. Since the distance between the masses is $L_0 + x_2 - x_1$, the required minimum, i.e., the maximum of the distance $x_1 - x_2$, is reached when $v_1 = v_2$, and into this condition we must substitute the expressions for v_1 and v_2 from solution (3.3) for step 2. From this we find the unique time t_1 at which the masses approach to within the least distance:

$$t_1 = \tau_1 + \theta, \quad \theta = \frac{(f_1 - 1)\tau_1}{f + 1 + \mu(f - \nu)} \tag{3.16}$$

We substitute (3.16) into the formulae for x_1 and x_2 from (3.3) and equate the maximum distance $x_1(t_1) - x_2(t_1)$ to the given quantity η of condition (2.3). We obtain

$$\eta = \frac{gk_1\tau_1^2}{2}\,\frac{(f_1-1)(f_1+f+\mu f-\mu\nu)}{f+1+\mu(f-\nu)} \tag{3.17}$$

These formulae enable one to solve different versions of problems of the motion of a two-mass system. Let us consider a few such problems.

1. Suppose we are given the parameters of the system (the masses m_1 and m_2 and the coefficients k_1 and k_2), the total displacement ξ and the admissible range of relative motion η, as well as the total duration T of the cycle of motion. It is required to find the other characteristics of the motion, f, f_1, f_4 and τ_i (i = 1, 2, 3, 4). Equations (3.15) and (3.17), into which we must substitute expression (3.14) for $\tau 1$, form a system of three algebraic equations for the three unknowns f, f_1 and f_4. This system is rather cumbersome and depends on several parameters, and its solution must satisfy inequalities (3.10). The only simplification is that any of Eqs (3.15) is linear in f_4, so that the system reduces to a system of two algebraic equations for f and f_1. Having solved this system by a suitable numerical method, one can determine the durations of the steps τ_i (i = 1, 2, 3, 4) from formulae (3.13), (3.14).

2. Another formulation of the problem assumes that, besides the system parameters m_1, m_2, k_1, k_2, the parameters of the first step of motion τ_1 and f_1 are also given, as well as η. It is required to find the other parameters of the motion ($f, f_4, \tau_2, \tau_3, \tau_4, T, \xi$). This problem is easily solved: from Eq. (3.17), which is linear in f, we find f, and then, eliminating ξ from Eqs (3.15), we derive a linear equation for f_4. We determine f_4 and then also ξ. The durations of the time intervals $\tau_2, \tau_3, \tau_4, T$ are found from Eqs (3.13) and (3.14). In this case all the unknown quantities are uniquely determined in explicit form. It is only necessary to verify inequalities (3.10).

3. The initial and boundary conditions (3.1) and (3.8) indicate that the motion just constructed can be repeated any number of times. In the process, the control force and velocity of motion will be varied periodically with period T, and the displacements of both bodies will receive equal increments ξ in each cycle of motion. It is therefore natural to try to maximize the average velocity of displacement of the system as a whole, which is

$$v = \xi/T \tag{3.18}$$

This problem will be investigated below.

4. OPTIMIZATION

We first fix all the parameters of the system (m_1, m_2, k_1, k_2) and the admissible range of relative motion η. We wish to find the parameters ξ, T, f, f_1, f_4 and τ_i (i = 1, 2, 3, 4) which maximize the average velocity of motion.

This problem will be solved with one simplifying assumption: the magnitude of the force F_2 in the second and third steps is infinite. This assumption is quite natural: these steps involve the main displacement of the mass m_2, and there is no upper limit on the magnitude of the propelling force $-F_2$ (see (2.4)). In that case, the motion in steps 2 and 3 reduces to an impulse of the finite magnitude.

By formulae (3.9) and (3.13), we have

$$f \to \infty,\quad \tau_2 \to 0,\quad \tau_3 \to 0,\quad f(\tau_2+\tau_3) \to \frac{(f_1-1)(f_4+\nu)}{\nu-1}\tau_1 = q \tag{4.1}$$

The quantity q is equal, apart from the factor m_1gk_1, to the magnitude of the impulse exchanged by masses m_1 and m_2 in the second and third steps of the motion.

It follows from (3.13) and (3.14), under condition (4.1), that

$$\tau_1 = \frac{(\nu-1)T}{f_1+\nu-2},\quad \tau_4 = \frac{(f_1-1)T}{f_1+\nu-2} \tag{4.2}$$

Relations (3.15) and (3.16) also simplify when $f \to \infty$, giving

$$\xi = \frac{gk_1\tau_1^2(f_1-1)[(\nu-1)^2-(f_4+1)(f_1-1)]}{2(\nu-1)^2} \tag{4.3}$$

$$\xi = \frac{gk_1\tau_1^2\mu(f_4+\nu)(f_1-1)^2}{2(\nu-1)^2},\quad \eta = \frac{gk_1\tau_1^2(f_1-1)}{2}$$

Eliminating ξ from the first two equations of (4.3), we obtain

$$f_4 = \frac{(\nu-1)^2 - (f_1-1)(1+\mu\nu)}{(f_1-1)(1+\mu)} \tag{4.4}$$

A direct check verifies that, if the inequalities $\mu < 1$ and $1 < f_1 < \nu$ of (3.9) and (3.10) are satisfied then f_4 of (4.4) satisfies the condition $f_4 > -1$ of (3.10). Substituting f_4 from (4.4) into the second equation of (4.3), we have

$$\xi = \frac{gk_1\tau_1^2\mu(f_1-1)(f_1+\nu-2)}{2(\nu-1)(1+\mu)} \tag{4.5}$$

To determine the average velocity v, we substitute ξ from (4.5) and T from the first equation of (4.2) into (3.18), obtaining

$$v = \frac{gk_1\tau_1\mu(f_1-1)}{2(1+\mu)} \tag{4.6}$$

Substituting the value of τ_1 obtained from the last relation of (4.3) into (4.6), we finally have

$$v = \left[\frac{gk_1\eta(f_1-1)}{2}\right]^{1/2} \frac{\mu}{1+\mu} \tag{4.7}$$

It follows from (4.7) that the velocity v increases monotonically as f_1 increases. The maximum v is achieved at the upper limit of (3.10) for f_1, i.e., in the limit as $f_1 \to \nu$. Now, using formulae (4.1)–(4.4) and (4.7), we obtain the optimum values of all the parameters:

$$\begin{gathered} f \to \infty, \quad f_1 \to \nu, \quad f_4 = \frac{\nu-2-\mu\nu}{1+\mu} \\ \tau_1 = \tau_4 = T/2, \quad T = 2\left[\frac{2\eta}{gk_1(\nu-1)}\right]^{1/2} \\ \xi = \frac{2\eta\mu}{1+\mu}, \quad v_{\max} = \left[\frac{gk_1\eta(\nu-1)}{2}\right]^{1/2} \frac{\mu}{1+\mu} \end{gathered} \tag{4.8}$$

Strictly speaking, the required maximum velocity $v_{\max}$ is unattainable, but it may be approximated as closely as desired as $f_1 \to \nu$.

Using relations (3.9), we transform formula (4.8) for $v_{\max}$ to dimensional variables:

$$v_{\max} = [g\eta m_1(m_2k_2 - m_1k_1)/2]^{1/2}(m_1+m_2)^{-1} \tag{4.9}$$

Let us investigate the influence of the coefficients of friction and the mass ratio on the maximum velocity of displacement. If the coefficients of friction lie within the limits

$$k^- \leqslant k_1 \leqslant k^+, \quad k^- \leqslant k_2 \leqslant k^+$$

where k^- and k^+ are given numbers, the maximum velocity is reached when $k_1 = k^-$, $k_2 = k^+$.

For fixed coefficients of friction, the function $v_{\max}$ in (4.9) may be expressed as a function of the mass ratio $\mu = m_2/m_1$ as follows:

$$v_{\max} = [g\eta\varphi(\mu)/2]^{1/2}, \quad \varphi(\mu) = \mu(k_2 - \mu k_1)(1+\mu)^{-2} \tag{4.10}$$

The function $\varphi(\mu)$ has a single maximum in the interval (0, 10), when

$$\mu = k_2(2k_1+k_2)^{-1} < 1 \tag{4.11}$$

The corresponding maximum velocity (4.10) is

$$v^*_{\max} = (g\eta)^{1/2} k_2[8(k_1 + k_2)]^{-1/2} \tag{4.12}$$

In the special case when the masses m_1 and m_2 have the same coefficients of friction, i.e., $k_1 = k_2 = k$, it follows from (4.8)–(4.12) that

$$\mu = m_1 / m_2 = 1/3, \quad \nu = 3, \quad f_1 = 3, \quad f_4 = 0$$

$$T = 2\eta^{1/2}(gk)^{-1/2}, \quad \xi = \eta/2, \quad v^*_{\max} = (g\eta k)^{1/2}/4 \tag{4.13}$$

We have thus obtained simple, readily understood relations for the optimum motion. Naturally, in order to increase the average velocity of motion, the coefficient of friction k_2 of the primary mass should be increased and that of the secondary mass, k_1, decreased. We also have a simple formula (4.11) for the mass ratio. In the case of equal coefficients of friction, the secondary mass must be one third of the primary mass, and the total displacement ξ per cycle of optimum motion turns out to be half of the admissible range η of relative motion. We recall, however, that all these conclusions were obtained with the control force subject to no restrictions; in one step of the motion, in fact, the control force was even assumed to be infinite (impulsive). By imposing restrictions on the control force and solving the resulting optimization problems numerically, one obtains more realistic – but less intuitive – results.

The two-mass system considered above is of interest, as a simple mechanical model capable of moving on a plane in the presence of friction, as a result of internal control forces. The motion of such a system will be more efficient if energy regeneration is allowed, that is, a spring is incorporated.

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REFERENCES

1. DOBROLYUBOV, A. I., *Travelling Deformation Waves*. Nauka i Tekhnika, Minsk, 1987.
2. CHERNOUS'KO, F. L., The wavelike motion of a multilink system on a horizontal plane. *Prikl. Mat. Mekh.*, 2000, **64**, 4, 518–531.

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